Solution 10

1. Let E be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in E if it is bounded at a single point, that is, if there are $x_0 \in E$ and constant M such that $|f(x_0)| \le M$ for all f in this family.

   Solution. By equicontinuity, for $\varepsilon = 1$, there is some δ_0 such that $|f(x) - f(y)| \le 1$ whenever $|x - y| \le \delta_0$. Let $B_R(x_0)$ a ball containing E. Then $|x - x_0| \le R$ for all $x \in E$. We can find $x_0, \cdots, x_n = x$ where $n\delta_0 \le R \le (n+1)\delta_0$ so that $|x_{n+1} - x_n| \le \delta_0$. It follows that
   $$|f(x) - f(x_0)| \le \sum_{j=0}^{n-1} |f(x_{j+1} - f(x_j)| \le n \le \frac{R}{\delta_0}.$$
   Therefore,
   $$|f(x)| \le |f(x_0)| + n + 1 \le M + \frac{R}{\delta_0} \quad \forall x \in E,\ \forall f \in \mathcal{F}.$$

2. Let $\{f_n\}$ be a sequence of bounded functions in $[0,1]$ and let F_n be
   $$F_n(x) = \int_0^x f_n(t)dt.$$
   (a) Show that the sequence $\{F_n\}$ has a convergent subsequence provided there is some M such that $\|f_n\|_\infty \le M$, for all n.

   (b) Show that the conclusion in (a) holds when boundedness is replaced by the weaker condition: There is some K such that
   $$\int_0^1 |f_n|^2 \le K, \quad \forall n.$$

   Solution.

   (a) Since $|F_n| \le \int_0^x |f_n(t)|dt \le M$, and $|F_n(x) - F_n(y)| \le \int_y^x |f_n(t)|dt \le |x-y|M$, $\{F_n\}$ is uniformly bounded and equicontinuous. Then it follows from Arzela-Ascoli theorem that $\{F_n\}$ is sequentially compact.

   (b) It follows from the Cauchy-Schwarz inequality that
   $$|F_n(x) - F_n(y)| \le \int_y^x |f_n(t)|dt \le \left(\int_y^x 1^2 dt\right)^{1/2} \left(\int_y^x |f_n(t)|^2 dt\right)^{1/2} \le \sqrt{K}\sqrt{|x-y|}.$$
   Similarly one can show that $\{F_n\}$ is uniformly bounded. Then apply Arzela-Ascoli theorem.

3. Prove that the set consisting of all functions G of the form
   $$G(x) = \sin^2 x + \int_0^x \frac{g(y)}{1 + g^2(y)}\, dy\ ,$$
   where $g \in C[0,1]$ is precompact in $C[0,1]$.

   Solution. Straightforward to check $\|G\|_{L^\infty} \le 2$ and $\|G'\|_{L^\infty} \le 3$. By Ascoli's Theorem this set is precompact.

4. Let $K \in C([a,b] \times [a,b])$ and define the operator T by
   $$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$

(a) Show that T maps $C[a,b]$ to itself.

(b) Show that whenever $\{f_n\}$ is a bounded sequence in $C[a,b]$, $\{Tf_n\}$ contains a convergent subsequence.

Solution.

(a) Since $K \in C([a,b]\times[a,b])$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(x,y) - K(x',y)| < \varepsilon$, whenever $|x-x'| < \delta$. Then for $x, x' \in [a,b], |x-x'| < \delta$, one has

$$|(Tf)(x) - (Tf)(x')| \leq \int_a^b |K(x,y) - K(x',y)||f(y)|dy \leq |a-b|\|f\|_\infty \varepsilon.$$

Hence $Tf \in C[a,b]$.

(b) Suppose $\sup_n \|f_n\|_\infty \leq M < \infty$. It follows from the proof of (a) that δ can be taken independent of n. Hence $\{f_n\}$ is equicontinuous. Furthermore, since $|(Tf_n)(x)| \leq \int_a^b |K(x,y)||f_n(y)|dy \leq M(b-a)\|K\|_\infty$, $\{f_n\}$ is uniformly bounded. Then it follows from Arzela-Ascoli theorem that $\{Tf_n\}$ contains a convergent subsequence.

5. Let f be a bounded, uniformly continuous function on $\mathbb{R}$. Let $f_a(x) = f(x-a)$. Show that there exists a sequence of unit intervals $I_k = [n_k, n_k+1]$, $n_k \to \infty$, such that $\{f_{n_k}\}$ converges uniformly on $[0,1]$.

   Solution. Each f_n is defined on $[0,1]$. Since f is bounded and uniformly continuous on $\mathbb{R}$, $\{f_n\}$ is uniformly bounded and equivcontinuous. Apply Ascoli's theorem it contains a uniformly convergent subsequence.

   Note. The lesson is, if you keep watching dramas in TVB every evening, soon you find some new one resembling an old one.

6. Optional. A bump function is a smooth function φ in $\mathbb{R}^2$ which is positive in the unit disk, vanishing outside the ball, and satisfies $\iint_{\mathbb{R}^2} \varphi(x)\, dA(x) = 1$. Let f be a continuous function defined in an open set containing $\overline{G}$ where G is bounded and open in $\mathbb{R}^2$. For small $\varepsilon > 0$, define

$$f_\varepsilon(x) = \frac{1}{\varepsilon^2}\iint_{\mathbb{R}^2} \varphi\left(\frac{y-x}{\varepsilon}\right) f(y)\, dA(y)\ .$$

   Show that f_ε is $C^\infty(\overline{G})$ and tends to f uniformly as $\varepsilon \to 0$.

   Note. This property has been used in the proof of Cauchy-Peano theorem.

   Solution. Since φ is smooth and the integration is in fact over a bounded set, for $j = 1, 2$,

$$\frac{\partial f_\varepsilon}{\partial x_j}(x) = \frac{-1}{\pi\varepsilon^3}\iint_{\mathbb{R}^2} \frac{\partial \varphi}{\partial x_j}\left(\frac{y-x}{\varepsilon}\right) f(y)\, dA(y)\ .$$

   By consecutive differentiation, one sees that f_ε is smooth. Next, note that

$$\frac{1}{\varepsilon^2}\iint_{\mathbb{R}^2} \varphi\left(\frac{y-x}{\varepsilon}\right) dA(y) = 1.$$

For, letting B be the ball $\{y : \ |y - x| \le \varepsilon\}$,

$$\begin{aligned}\iint_{\mathbb{R}^2} \varphi\left(\frac{y-x}{\varepsilon}\right) dA(y) &= \iint_B \varphi\left(\frac{y-x}{\varepsilon}\right) dA(y) \\ &= \iint_{\mathbb{R}^2} \varphi\left(\frac{y}{\varepsilon}\right) dA(y) \\ &= \varepsilon^2 \iint_{\mathbb{R}^2} \varphi(z)\, dA(z) \\ &= \varepsilon^2 \ .\end{aligned}$$

Now, for $\varepsilon' > 0$, there is some δ such that $|f(y) - f(x)| < \varepsilon'$ when $|x - y| < \delta, x, y \in \overline{G}$. For $\varepsilon \in (0, \delta)$,

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &= \left|\frac{1}{\varepsilon^2} \iint_B \varphi\left(\frac{y-x}{\varepsilon}\right) (f(y) - f(x))\, dA(y)\right| \\ &\le \frac{1}{\varepsilon^2} \iint_B \varphi\left(\frac{y-x}{\varepsilon}\right) |f(y) - f(x)|\, dA(y) \\ &< \varepsilon' \ ,\end{aligned}$$

hence $f_\varepsilon \to f$ uniformly as $\varepsilon \to 0$.

7. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $\mathbb{R}$ (you may draw a table):

   (a) $A = \{n/2^m : \ n, m \in \mathbb{Z}\}$,
   (b) B, all irrational numbers,
   (c) $C = \{0, 1, 1/2, 1/3, \cdots\}$,
   (d) $D = \{1, 1/2, 1/3, \cdots\}$,
   (e) $E = \{x : \ x^2 + 3x - 6 = 0\ \}$,
   (f) $F = \cup_k (k, k+1), k \in \mathbb{N}$,

Solution. (a) A is dense, not open, not nowhere dense, of first category and not residual.

(b) B is dense, not open, not nowhere dense, of second category and residual.

(c) C is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

(d) D is not dense, not open (not closed), nowhere dense, of first category and not residual.

(e) E is the finite set $\{(-3 + \sqrt{33})/2, (-3 - \sqrt{33})/2\}$. It is not dense, not open (closed in fact), nowhere dense, of first category and not residual.

(e) F is dense, open, not nowhere dense, of second category and residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
A	✓	✗	✗	✓	✗
B	✓	✗	✗	✗	✓
C	✗	✗	✓	✓	✗
D	✗	✗	✓	✓	✗
E	✗	✗	✓	✓	✗
F	✓	✓	✗	✗	✓

8. Determine which of the following sets are dense, open dense, nowhere dense, of first category and residual in $C[0,1]$ (you may draw a table):

   (a) $\mathcal{A}$, all polynomials whose coefficients are rational numbers,

   (b) $\mathcal{B}$, all polynomials,

   (c) $\mathcal{C} = \{f : \int_0^1 f(x)dx \neq 0\}$,

   (d) $\mathcal{D} = \{f : f(1/2) = 1 \}$.

   Solution. (a) $\mathcal{A}$ is dense (and countable too), not open, not nowhere dense, of first category, and not residual.

   (b) $\mathcal{B}$ is dense (and uncountable), not open, not nowhere dense, of first category and not residual. ($\mathcal{B}$ can be expressed as the countable union of P_n where P_n is the set of all polynomials of degree not exceeding n. Each P_n is closed and nonwhere dense.)

   (c) $\mathcal{C}$ is dense, open, not nowhere dense, of second category, and residual.

   (d) $\mathcal{D}$ is not dense, not open (closed in fact), nowhere dense, of first category, and not residual.

Sets	Dense	Open dense	Nowhere dense	First category	Residual
$\mathcal{A}$	✓	✗	✗	✓	✗
$\mathcal{B}$	✓	✗	✗	✓	✗
$\mathcal{C}$	✓	✓	✗	✗	✓
$\mathcal{D}$	✗	✗	✓	✓	✗